A power-law alternative to simple exponential decay for modeling mass loss due to decomposition. (DRAFT)

Kevin R. Gehringer[1]

Biometrics Northwest LLC, Redmond, WA, 98053

Abstract

A new decomposition model $\dot{x}(t) = \alpha[x(t)]^g,\ x(0) = X_0$ representing litter mass remaining after time t is developed. The new decomposition model was derived by evaluating the assumptions and limitations of the commonly used simple exponential decay model to derive a power-law generalization as a nonlinear differential equation. A closed form or integrated solution of the new model is also derived for completeness. The power-law model is shown to perform better than the simple exponential decay model for long term projections, and it performed as well as a two compartment, double exponential decay decomposition model representing fast and slow decomposition fractions but without the need to assume a second compartment. Needle litter data from a litter bag study in the West Twin Creek watershed of the Hoh River valley, located on the western side of Olympic National Park in Washington State, USA were used to fit the models and assess their performance.

Keywords: Power-law model, simple exponential decay model, multi-comapartment model, needle litter, decomposition

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1. Introduction

Simple exponential decay, $x(t) = X_0 e^{-kt}$, is commonly used to model the decomposition process for a single type of material, e.g., needle and leaf litter, twigs and small branches, large branches, or logs of different size classes, etc., or for a single type of

Email address: `krg@biometricsnw.com` (Kevin R. Gehringer)

material under different physical conditions, e.g., different elevations, climatic conditions, etc., where $x(t)$ represents the litter mass remaining after time t, given an initial mass $x(0) = X_0 > 0$ and a characteristic rate of decay $k > 0$ (Olson, 1963; Edmonds, 1984; Perry, 1994; Edmonds and Thomas, 1995; Schlesinger, 1997). Each type of material and different physical condition may be considered as a separate compartment in a model representing the total mass remaining, where each compartment i has a different characteristic rate of decay k_i, initial mass $x_i(0) = X_{0i} > 0$, and corresponding simple exponential decay submodel $x_i(t) = X_{0i}e^{-k_i t}$ representing the remaining mass in that compartment after time t. For N different material classes or physical conditions the total mass remaining is represented by the sum of the individual exponential decay submodels for each compartment as in Equation 1.

$$x(t) = \sum_{i=1}^{N} x_i(t) = \sum_{i=1}^{N} X_{0i}e^{-k_i t} \tag{1}$$

Simple exponential decay decomposition models are frequently expressed as the proportion of the initial litter mass remaining after time t, as in Equation 2 for a single compartment model

$$y(t) = \frac{x(t)}{X_0} = e^{-kt} \tag{2}$$

or Equation 3 for a multi-compartment exponential decay model

$$y(t) = \frac{x(t)}{\sum_{i=1}^{N} X_{0i}} = \frac{\sum_{i=1}^{N} x_i(t)}{\sum_{j=1}^{N} X_{0j}} = \sum_{i=1}^{N} \frac{X_{0i}}{\sum_{j=1}^{N} X_{0j}} e^{-k_i t} = \sum_{i=1}^{N} p_{0i}e^{-k_i t} = \sum_{i=1}^{N} p_{0i}y_i(t) \tag{3}$$

where $y_i(t) = e^{-k_i t}$ and $\sum_{i=1}^{N} p_{0i} = 1$.

Simple exponential decay decomposition models may also be described as first order, linear differential equations with constant coefficients (Boyce and DiPrima, 1986) as in Equation 4 and Equation 5 for single compartment models

$$\dot{x}(t) = -kx(t),\ x(0) = X_0 > 0 \tag{4}$$

$$\dot{y}(t) = -ky(t),\ y(0) = 1 \tag{5}$$

or Equation 6 and Equation 7 for multi-compartment models.

$$\dot{x}(t) = \sum_{i=1}^{N} \dot{x}_i(t) = \sum_{i=1}^{N} -k_i x_i(t),\ x_i(0) = X_{0i} > 0,\ i = 1, \ldots, N \tag{6}$$

$$\dot{y}(t) = \sum_{i=1}^{N} p_{0i}\dot{y}_i(t) = \sum_{i=1}^{N} p_{0i}\left(-k_i y_i(t)\right),\, y_i(0) = 1,\, i = 1, \ldots, N \tag{7}$$

Single compartment simple exponential decay models for decomposition are almost universally used to estimate long term average decomposition rates for forest litter (Olson, 1963; Edmonds, 1984; Harmon et al., 1990; Perry, 1994; Edmonds and Thomas, 1995; Schlesinger, 1997). This application of the simple exponential decay model to the decomposition process is, however, not justified: the simple exponential decay model is appropriate only for short time intervals relative to the rate of the decomposition process, e.g., days, months or years depending on the environmental conditions and the type of litter being decomposed (Olson, 1963; Perry, 1994). The poor performance of the simple exponential decay model when estimating long term decomposition rates has been reported by various authors (Edmonds, 1984; Harmon et al., 1990; Perry, 1994; Edmonds and Thomas, 1995). The primary problem with the simple exponential decay models is that they do not account for the fact that as woody debris or litter decomposes the remaining mass is more difficult to decompose (Perry, 1994; Schlesinger, 1997), and the rate of decomposition slows over time as the chemical content of the remaining mass becomes more difficult for fungi and bacteria to decompose. This leads to a too rapid decrease of the remaining litter mass, resulting in an overestimation of the litter mass lost. Further, the characteristic rate k for a single compartment simple exponential decomposition model is not constant; estimates of characteristic rates decline as the time interval of decomposition increases (Edmonds, 1984; Harmon et al., 1990; Edmonds and Thomas, 1995).

Multi-compartment exponential decay decomposition models have been used in an attempt to resolve some of the problems that occur when using a single compartment simple exponential decay model. Bunnell and Tait (1974) proposed the "double exponential" two compartment model consisting of a fast decay component and a slow decay component in Equation 8,

$$y(t) = F_{\text{slow}} e^{-k_{\text{slow}} t} + (1 - F_{\text{slow}}) e^{-k_{\text{fast}} t} \tag{8}$$

where F_{slow} is the fraction of the mass subject to slow decomposition and k_{slow} and k_{fast} are the characteristic rates for the slow and fast decomposition fractions, respectively. With this model the fast and slow decomposition compartments are assumed to exist and their fractions and characteristic rates are estimated via nonlinear regression.

A new single compartment decomposition model for the decrease in litter mass over time is developed. The new model is a straightforward extension of the simple exponential decay decomposition model to a power-law representation and imposes no additional modeling assumptions. The new model is applied to needle decomposition data obtained from litter bag experiments (Edmonds and Thomas, 1995) and is compared with the simple exponential decay decomposition model and the two compartment double exponential decay fast/slow model using the same data. The new model is shown to provide an excellent fit to the data, overcoming the primary limitations of the simple exponential decay decomposition model, without the assumption of a second compartment, as was necessary for the two compartment double exponential decay fast/slow model to obtain a reasonable fit to the data. The new power-law decomposition model also appears to have the highly desirable property of producing consistent parameter estimates with the addition of new data over time: the new data fine-tunes the parameter values and confidence intervals rather than causing significant changes in the parameter values, as happens with the simple exponential decay model.

2. Methods

The simple exponential decay decomposition model for a uniform material has several features that recommend it. First, the current rate of decomposition is related to the current litter mass, a physically and biologically reasonable assumption. This relationship is made concise by assuming that the rate of decomposition is proportional to the current litter mass as in Equation 9.

$$\dot{x}(t) \propto x(t). \tag{9}$$

Recognizing that the amount of litter may only decrease then implies a negative constant of proportionality, yielding the simple exponential decay decomposition model

$$\dot{x}(t) = -kx(t),\, k > 0 \tag{10}$$

subject to an initial condition $x(0) = X_0 > 0$ (Boyce and Diprima, 1986). The simple exponential decay decomposition model makes a minimum number of assumptions concerning the physical and biological processes involved in the decrease of litter mass due to decomposition and the mathematical properties of $x(t)$. Specifically the assumptions are:

(1) All factors influencing the rate of decomposition at a particular point in time are integrated, or averaged, into the decrease in litter mass up to that time.

(2) The decomposition rate is proportional to the current litter mass.

(3) The decomposition process may only decrease the litter mass.

(4) The function $x(t)$ must be continuously differentiable.

The first assumption is necessary to even consider modeling the decomposition process using the remaining litter mass. The third and fourth assumptions are necessary to obtain the necessary behavior for the decomposition model. The second assumption, however, is somewhat arbitrary and it provides the starting point for development of the new decomposition model. As originally stated, the second assumption was simply that the current rate of decomposition was related to the current litter mass; the exact relationship was not specified. Assuming that the rate of decomposition was proportional to the current litter mass provided a concise mathematical relationship, but only one of many possible relationships.

What properties should the relationship between the rate of decomposition and the current litter mass have? First, the rate of decomposition should decrease more rapidly than the simple exponential decay model allows as the amount of litter mass remaining decreases. This will overcome the too rapid decrease in mass produced by the simple exponential decay model. Second, and this is arbitrary, the new model should be a straightforward extension or generalization of the simple exponential decay model. The catabolic loss term βW^g from a Bertalanffy growth law (Vanclay, 1994) in Equation 11

$$\dot{W} = \alpha W - \beta W^g,\ \alpha > 0, \beta > 0 \tag{11}$$

provides the desired generalization of the simple exponential decay decomposition model, yielding the power-law relationship in Equation 12 as the new model for the decrease in litter mass over time.

$$\dot{x}(t) = -\alpha[x(t)]^g,\ x(0) = X_0 > 0 \tag{12}$$

This model meets both of the stated conditions and is also a decay model provided $\alpha > 0$ and $g \geq 1$. This rate equation model may be analytically integrated to produce

Equation 13

$$x(t) = (\alpha' t + {X_0}^{1-g})^{\frac{1}{1-g}}, \tag{13}$$

with $\alpha' = -\alpha(1-g)$ to give a closed form equation for the remaining litter mass analogous to $X_0 e^{-kt}$ for simple exponential decay. As presented, Equation 13 for the remaining litter mass does not look like a decreasing function, but using the relationships imposed on α and g to obtain a decay model Equation 13 can be rewritten to obtain Equation 14 where $g - 1 > 0$ and $\alpha' > 0$, an equation that obviously decreases to zero as t increases.

$$x(t) = X_0 \left(\frac{1}{X_0^{g-1} \alpha' t + 1} \right)^{\frac{1}{g-1}} \tag{14}$$

Unfortunately, the closed from solution for the power-law decomposition model has a singularity for $g = 1$, the value of the exponent that produces the simple exponential decay model. This is a deficiency of the closed form solution introduced by the separation of variables technique used to analytically solve the differential equation and derive the solution. The differential equation may be solved numerically for $g = 1$, and a numerical solution approach is strongly recommended, particularly when estimating the model parameters α and g from data. For completeness, the new decomposition model expressed as a proportion of the initial litter mass is presented as a differential equation and in closed form in Equation 15 and Equation 16, respectively.

$$\dot{y}(t) = -\alpha[y(t)]^g,\ y(0) = 1 \tag{15}$$

$$y(t) = (\alpha' t + 1)^{\frac{1}{1-g}} \tag{16}$$

2.1. Data

Green needle decomposition data were obtained from a litter bag study conducted in the West Twin Creek watershed in the Hoh River Valley on the western side of Olympic National Park, Washington, USA (Edmonds and Thomas, 1995). Elevations in the watershed range from 180 m to 850 m. The watershed contains portions of the western hemlock (*Tsuga Heterophylla*) zone in the lower elevations and the Pacific silver fir (*Abies amabilis*) zone in the upper elevations (Franklin and Dyrness, 1988). Mean temperatures for January and July are 4 °C and 16 °C, respectively, with a mean annual rainfall of approximately 350 cm (Edmonds and Thomas, 1995).

Table 1: Mean, standard deviation (SD), and median values computed using all seven needle litter bags collected at each time for the proportion of Pacific silver fir litter mass remaining.

i	Years	Mean	SD	Median
0	0.00	1.00	—-	1.00
1	0.32	0.91	0.02	0.92
2	1.00	0.73	0.03	0.73
3	3.08	0.51	0.07	0.52
4	5.08	0.39	0.15	0.35

Green needles from Pacific silver fir were collected, air dried, weighed, and placed into polyester litter bags with a 1 mm mesh. Litter bags containing the dried Pacific silver fir needles were then randomly placed in a 0.1 ha circular plot at an elevation of 275 m. Seven litter bags were collected after 4, 12, 37, and 61 months (0.32, 1.00, 3.08, and 5.08 years) (Edmonds and Thomas, 1995). The mean values and standard deviations for the proportion of the initial needle litter mass remaining for the seven litter bag samples from each collection time i are presented in Table 1. Note that the standard deviation increases as the proportion of litter mass decreases over time, indicating that the range of observed values is also increasing over time.

2.2. Model comparison and parameter estimation

The proportional model of litter mass remaining $y(t)$ after time t, measured in years, were used to perform the parameter estimation and analysis for the single compartment simple exponential decay and power-law models, and the two compartment double exponential decay fast/slow decomposition model. All three models were treated as nonlinear models for calibration and no model or data transformations were used. For model parameter estimation, each of the $j = 1, 2, \ldots, 7$ samples taken at time t_i for was assumed to represent single trajectory having an initial time $t_0 = 0$ and litter mass proportion $y_0 = 1$. All of the samples therefore satisfy the assumption that decomposition can only decrease litter mass over time.

Parameters for the simple exponential decay and two compartment double exponential decay fast/slow models were estimated directly using the time series data and nonlinear least squares to obtain estimated parameters $\hat{k}$ for the simple exponential decay model and $\hat{F}_{\text{slow}}$, $\hat{k}_{\text{slow}}$ and $\hat{k}_{\text{fast}}$ for the two compartment double exponential decay fast/slow model. Parameter values for the power-law model were estimated using the differential equation power-law model formulation and nonlinear least squares in three steps. First, linear decay rates Δy_{ij} and average values for the proportion of litter mass remaining $\bar{y}_{ij}$ were computed for each time interval $[t_0, t_i]$ and replication j using Equation 17 and Equation 18, respectively.

$$\Delta y_{ij} = \frac{y_{ij} - y_0}{t_i - t_0}, i = 1, 2, 3, 4, j = 1, 2, \ldots, 7 \tag{17}$$

$$\bar{y}_{ij} = \frac{y_{ij} - y_0}{2}, i = 1, 2, 3, 4, j = 1, 2, \ldots, 7 \tag{18}$$

Second, the linear decay rates Δy_{ij} were assigned to the left hand side in Equation 15 with the average values $\bar{y}_{ij}$ approximating the proportion of litter mass remaining for the right hand side for each sample i and replication j. Initial parameter values of $\alpha = 0$ and $g = 0$, a constant model, were used to initialize the nonlinear least squares procedure to minimize Equation 19 and obtain the estimated parameter values $\hat{\alpha}$ and $\hat{g}$.

$$f_{\text{rate}}(\alpha, g) = \sum_{j=1}^{7} \sum_{i=1}^{4} (\Delta y_{ij} - \alpha[\bar{y}_{ij}]^g)^2 \tag{19}$$

Third, a shooting method was used to solve the power-law differential equation for each time interval $[t_0, t_i]$ and replication j obtaining values $\hat{y}_{ij}$, and then nonlinear least squares was used to estimate the final parameters $\hat{\alpha}_f$ and $\hat{g}_f$ for the power-law model by minimizing Equation 20, using the estimated parameter values $\hat{\alpha}$ and $\hat{g}$ from the first step as starting values for the least squares optimization.

$$f_{\text{shooting}}(\alpha, g) = \sum_{j=1}^{7} \sum_{j=1}^{4} (y_{ij} - \hat{y}_{ij})^2 \tag{20}$$

3. Results

Estimated values for the parameters of the three decomposition models with 95% confidence intervals are presented in Table 2. The value obtained for $\hat{k}$, the characteristic rate of the simple exponential decay model, agrees with the value obtained by Edmonds and Thomas (1995) for these data, and the order of magnitude difference between the

Table 2: Estimated parameter values for the three decomposition models.

			95% CI	
Model	Parameter	Estimate	Lower	Upper
Power-law	$\hat{\alpha}_f$	0.3004	0.1579	0.4429
	$\hat{g}_f$	1.5183	0.6297	2.4070
Fast/slow	$\hat{F}_{\text{slow}}$	0.6253	0.0486	1.2020
	$\hat{k}_{\text{slow}}$	0.0942	-0.0750	0.2635
	$\hat{k}_{\text{fast}}$	0.7803	-0.2486	1.8093
Simple	$\hat{k}$	0.2124	0.1928	0.2319

characteristic rates $\hat{k}_{\text{slow}}$ and $\hat{k}_{\text{fast}}$ is consistent with results reported by Harmon et al. (1990). The 95% confidence intervals for the estimates of the slow proportion $\hat{F}_{\text{slow}}$ and the characteristic rates $\hat{k}_{\text{slow}}$ and $\hat{k}_{\text{fast}}$ for the two compartment double exponential decay fast/slow model were all wider, relative to the magnitude of the estimated parameter values, than the confidence intervals for both of the other models, and all three confidence intervals for the two compartment double exponential decay fast/slow model contained zero. This was due to a slightly ill-conditioned covariance matrix for the estimated parameters, and may indicate an over-fit model relative to the amount of data.

Trajectories were computed for a 10 year projection using each of the three models and their estimated parameter values for comparison. The full trajectories and the data are presented in Figure 1. Predicted values at each data collection time and the observed mean values are also provided in Table 3 for comparison. The figure clearly demonstrates the problem with using simple exponential decay to model long term decomposition rates: the simple exponential decay model must overestimate the remaining litter mass for the earlier data while underestimating the remaining litter mass for the later data. This is clearly reflected in the predicted values for the Pacific silver fir litter mass proportion at each collection time, where the first and second predicted values are greater than their respective observed mean values by 0.0200 and 0.0787, and the fourth predicted value is less than its corresponding observed mean value by 0.0531. The crossover point for the simple exponential decay model occurs near the third collection time where the predicted value was 0.0138 greater than the observed mean value. Simple exponential decay models

Table 3: Predicted proportion of Pacific silver fir litter mass remaining for the power-law, fast/slow, and simple exponential decay models at each litter bag collection time with the mean and median values for comparison.

Years	Mean	Median	Power-law	Fast/slow	Simple
0.00	1.00	1.00	1.00	1.00	1.00
0.32	0.91	0.92	0.91	0.90	0.93
1.00	0.73	0.73	0.76	0.74	0.81
3.08	0.51	0.52	0.47	0.50	0.52
5.08	0.39	0.35	0.32	0.39	0.34

for decomposition are simply not flexible enough to fit long term decomposition data.

The power-law and two compartment double exponential decay fast/slow models both appear to fit the data well, passing through the centers of the observed Pacific silver fir litter mass proportion data for each collection time, having nearly identical trajectories until year 7, when the fast/slow trajectory becomes, and remains, smaller than the power-law trajectory, with the difference increasing, until the end of the 10 year projection. The predicted values for the Pacific silver fir litter mass proportion at each collection time further reinforce the quality of the model projections for the power-law and two compartment double exponential decay fast/slow models.

4. Discussion

A new power-law model for the decrease of litter mass over time due to decomposition has been proposed. The power-law model outperforms the commonly used simple exponential decay decomposition model, providing a high degree of fidelity to data over long time intervals, overcoming the most significant problem with simple exponential decay models of decomposition. The new power-law model also performed very well when compared with a two compartment double exponential decay fast/slow decomposition model (Bunnell and Tait, 1974) that has been recommended for wide use (Harmon et al., 1990). The proposed power-law decomposition model introduces no new assumptions, relative to the simple exponential decay model, changing only the mathematical form of the relationship between the current rate of decomposition and the current litter mass,

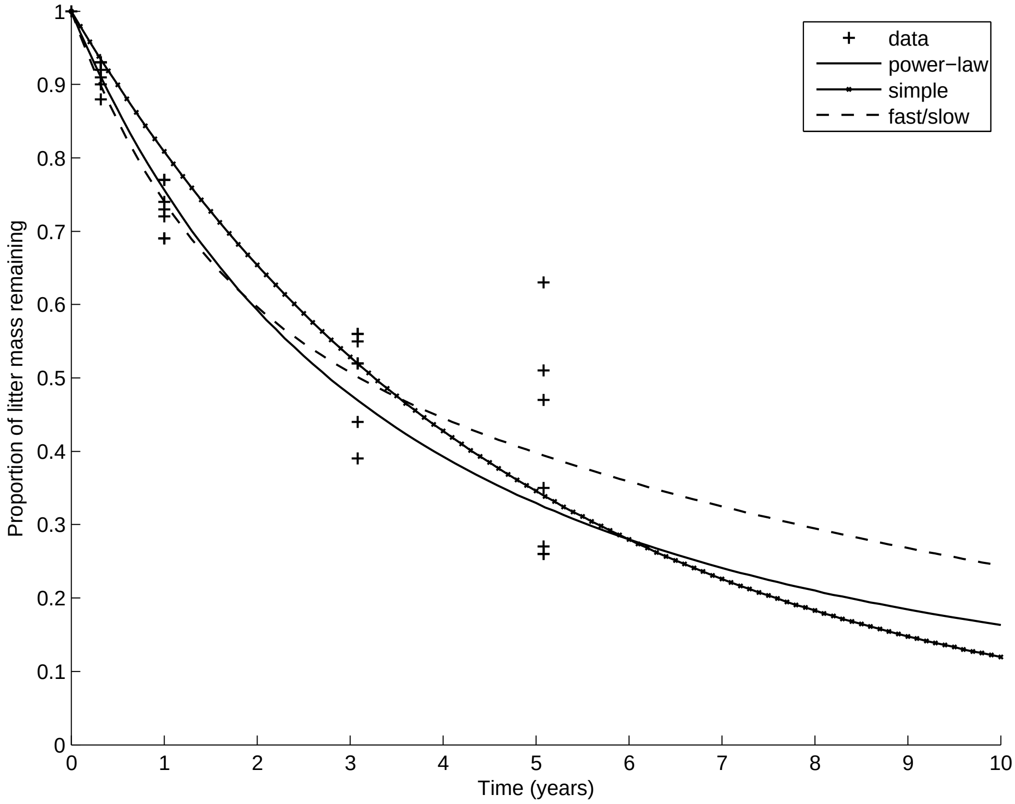

Figure 1: Data and estimated model trajectories for the proportion of Pacific silver fir needle mass remaining.

unlike the two compartment double exponential decay fast/slow decomposition model which assumes the existence of a second compartment in the decomposition process for uniform litter.

4.1. Single vs. multi-compartment models for uniform litter

The two-compartment double exponential decay fast/slow fraction model has been recommended as a replacement for the simple exponential decay decomposition model (Bunnell and Tait, 1974). Harmon et al. (1990) used this fast/slow fraction two compartment model with good success in an analysis of leaf litter decomposition rates, stating that the "flexibility of the double exponential model to fit the early as well as the later phases of litter decay warrant its wider use." Critical to using a model that assumes multiple compartments is that the compartments actually exist at the time scales of interest. Further, each compartment should be separately identifiable *a priori*, rather than assumed to be present and inferred *a posteriori*, after estimating model parameters. Otherwise any improvements in model fit may simply be a regression effect caused by the addition of more terms to a model and the consequent over fitting of the model when estimating parameters from data. In all likelihood, the two compartment fast/slow model fits the data well due to a regression effect, as it is highly unlikely that there is more than one identifiable compartment involved in needle or leaf litter decay over the time intervals of interest: several months to years.

The two-compartment double exponential decay fast/slow model suffers from the same problems as the simple exponential decay model, including decreasing parameter value estimates over longer time periods. Consider the two compartment fast/slow model and the addition of new data over longer and longer time intervals. The mass of the leaf litter continues to decrease, and, as is the case for a single compartment simple exponential decay model, the parameters of the fast/slow model also, necessarily, decrease with each new addition of data as the parameter estimation procedure attempts to balance the earlier data with the later data, leading to significantly different estimated parameter values over time. For the simple exponential decay decomposition model, this leads to a poor fit to the longer term data, with the model overestimating litter mass remaining for earlier times and underestimating it for later times. For the two-comapartment double exponential decay fast/slow decomposition model, good agreement with the addition of new data is still obtained, due to the flexibility of having the second exponential decay term, but the parameters must be estimated anew after each addition of data, severely limiting the predictive capabilities of the two-compartment double exponential decay fast/slow decomposition model for uses that extrapolate beyond the range of available data.

To demonstrate the regression effect when using the two-compartment double exponential decay fast/slow model for long term decomposition data we assumed that the power-law model using the estimated parameters for the Pacific silver fir needle litter, $\alpha_f = 0.3660$ and $g_f = 2.3365$, was the true underlying model for decomposition and considered two assessment scenarios. For both scenarios a 10 year trajectory was generated using the power-law model generating 10 uniformly spaced data points per year giving a total of 101 points when including the initial time $t = 0$. For the first scenario parameter values for the power-law model, two-compartment double exponential decay fast/slow model, and the simple exponential decay model were estimated using the data points from $t = 0$ to 2.5, 5.0, 7.5, and 10.0 years with no added variability. For the second scenario, 5% variability was added to the sampled data points used in the first scenario by randomly generating values r from a standard normal distribution $N(0, 1)$, and using Equation 21 to compute the trajectory with added variability.

$$y_{\text{var}} = y + 0.05 \times y \times r \tag{21}$$

Parameter values were then estimated using this trajectory for each of the three models and the same time intervals as in the first scenario.

For each scenario quality of fit assessments were made in two ways. First, we compared the estimated parameter values and their 95% confidence intervals across the four time intervals to assess the consistency of the parameter estimation process. Second, we compared the ten year trajectories produced by each estimated model for each of the scenarios graphically and by computing the integrated absolute error (IAE), using the formula in Equation 22, for the estimated trajectory, the input data trajectory, and the true trajectory,

$$\text{IAE} = \int_0^{10} |f(t) - g(t)|\, dt \tag{22}$$

where $f(t)$ and $g(t)$ represent the trajectories being compared, and a straightforward numerical integration technique was used to evaluate the integral. An IAE value of zero indicates that the two trajectories are identical, and IAE values greater than zero measure the area between the two trajectories indicating the degree of difference. The IAE has been used to compute an index comparing diameter distributions (Borders and Patterson, 1990; Reynolds et al., 1988), it has been used effectively to compare nonparametric estimates of probability density functions in one or more dimensions (Gehringer, 1990; Gehringer and Redner, 1992; Gehringer and Turnblom, 2014), and it may be the natural statistic for comparing probability density functions (Devroye and Gyorfi, 1985).

Estimated parameter values for the no-variability scenario are presented in Table 4 with the model and error trajectories presented in Figure 2 for the 2.5 year and 10.0 year data sets. Values for the power-law model were recovered essentially exactly, as expected. The estimated parameter values for the simple exponential decay model demonstrate the well understood behavior of declining values for the characteristic rate of decay as additional data are added. Further, each estimated characteristic rate is statistically significantly different from all of the others: there is no overlap in the 95% confidence intervals for any of the time intervals. Identical behavior was obtained for the three parameters of the two-compartment fast/slow model.

The trajectories in Figure 2 for the 2.5 year data set (top row) clearly indicate good early agreement with the data, and diverging trajectories for the simple exponential decay model and the two-compartment double exponential decay fast/slow model over the 10 year time interval, with the two-compartment double exponential decay fast/slow model performing better than simple exponential decay after 10 years. The trajectories in Figure 2 for the 10.0 year data set (bottom row) demonstrates very good agreement with the data for the two-compartment double exponential decay fast/slow model. A degradation in the early agreement for the simple exponential decay model to compensate for the additional data was also apparent, but the simple exponential decay model did obtain better agreement at 10 years where the error was reduced by a factor of two.

Estimated parameter values for the added variability scenario are presented in Table 5 with the model and error trajectories presented in Figure 3 for the 2.5 year and 10.0 year data sets. The addition of variability did not have a significant impact on the parameter values and confidence intervals for the simple exponential decay model, which

Table 4: Estimated parameters for four time intervals and no variability.

Model	Parameter	95% C.I.	2.5 years	5.0 years	7.5 years	10 years
Power-law		Lower	0.3004	0.3004	0.3004	0.3004
	$\hat{\alpha}_f$		0.3004	0.3004	0.3004	0.3004
		Upper	0.3004	0.3004	0.3004	0.3004
		Lower	1.5182	1.5181	1.5183	1.5183
	$\hat{g}_f$		1.5183	1.5182	1.5183	1.5183
		Upper	1.5183	1.5184	1.5184	1.5183
Fast/slow		Lower	0.7399	0.6268	0.5625	0.5150
	$\hat{F}_{\text{slow}}$		0.7556	0.6316	0.5676	0.5204
		Upper	0.7712	0.6364	0.5727	0.5257
		Lower	0.1724	0.1434	0.1286	0.1182
	$\hat{k}_{\text{slow}}$		0.1765	0.1445	0.1296	0.1193
		Upper	0.1806	0.1455	0.1307	0.1203
		Lower	0.6645	0.5590	0.5112	0.4795
	$\hat{k}_{\text{fast}}$		0.6860	0.5631	0.5152	0.4834
		Upper	0.7075	0.5673	0.5191	0.4873
Simple		Lower	0.2630	0.2411	0.2266	0.2166
	$\hat{k}$		0.2657	0.2444	0.2301	0.2202
		Upper	0.2685	0.2476	0.2335	0.2239

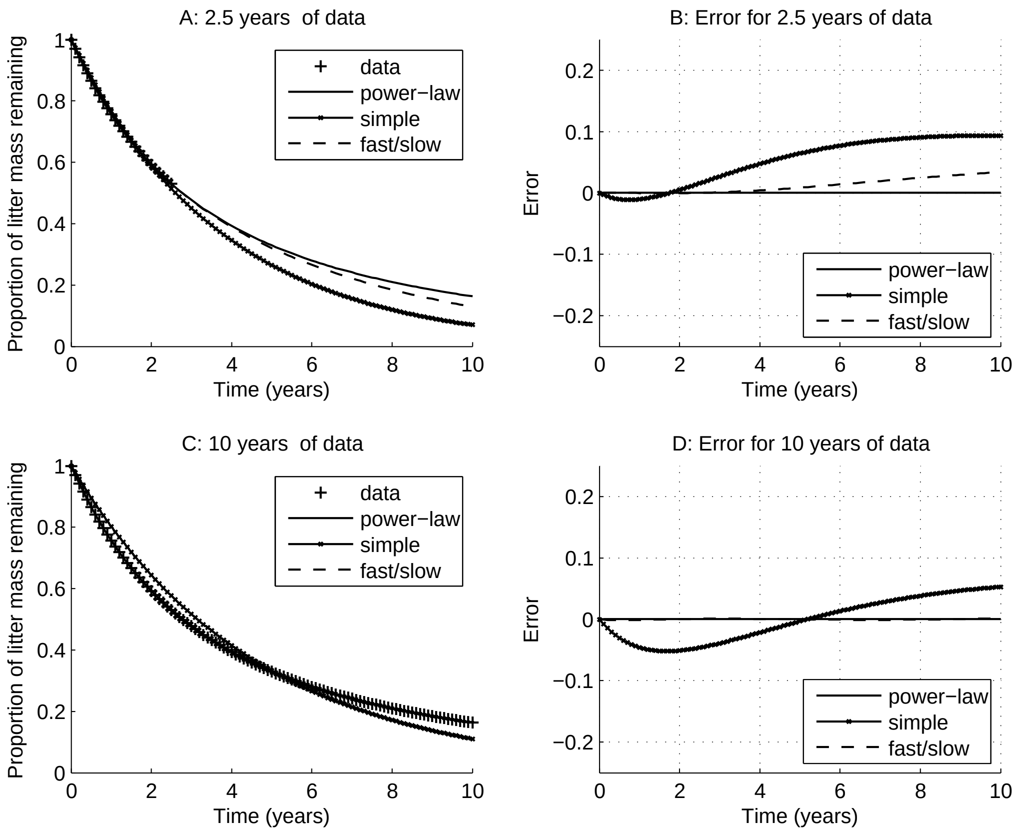

Figure 2: Data, estimated model, and error trajectories for the power-law, two-compartment fast/slow and simple exponential decay models for 2.5 and 10 years of data with no variability.

Table 5: Estimated parameters for four time intervals with 5% variability added.

Model	Parameter	95% C.I.	2.5 years	5.0 years	7.5 years	10 years
Power-law		Lower	0.1784	0.2464	0.2711	0.2803
	$\hat{\alpha}_f$		0.2652	0.2794	0.2918	0.2953
		Upper	0.3520	0.3123	0.3125	0.3104
		Lower	-0.0218	1.1065	1.3567	1.4275
	$\hat{g}_f$		1.3231	1.3595	1.4667	1.4912
		Upper	2.6679	1.6126	1.5768	1.5549
Fast/slow		Lower	0.9327	-1.7684	-0.2701	-0.0387
	$\hat{F}_{\text{slow}}$		0.9907	0.1088	0.1357	0.1885
		Upper	1.0487	1.9861	0.5415	0.4157
		Lower	0.1959	-1.8579	-0.2941	-0.0675
	$\hat{k}_{\text{slow}}$		0.2362	0.0000	0.0000	0.0350
		Upper	0.2764	1.8579	0.2941	0.1374
		Lower	-171.7421	-0.1297	0.1855	0.2443
	$\hat{k}_{\text{fast}}$		8.5029	0.2875	0.3017	0.3139
		Upper	188.7480	0.7046	0.4179	0.3834
Simple		Lower	0.2262	0.2325	0.2223	0.2137
	$\hat{k}$		0.2423	0.2392	0.2271	0.2180
		Upper	0.2583	0.2459	0.2320	0.2223

produced comparable estimated parameter values to the no added variability scenario and, essentially, behaved as they did for the no added variability scenario. The only difference from the no added variability scenario was that the 95% confidence intervals overlaped for the 2.5 and 5.0 year data sets in the added variability scenario, but the estimates of the characteristic rates $\hat{k}$ for these two data sets were outside the overlap regions and were still statistially significantly different. The trajectories for the simple exponential decay model and the added variability scenario in Figure 3 are also very similar to those from the no added variability scenario.

The addition of variability clearly impacted estimation of the parameters and 95% confidence intervals for the two-compartment double exponential decay fast/slow model

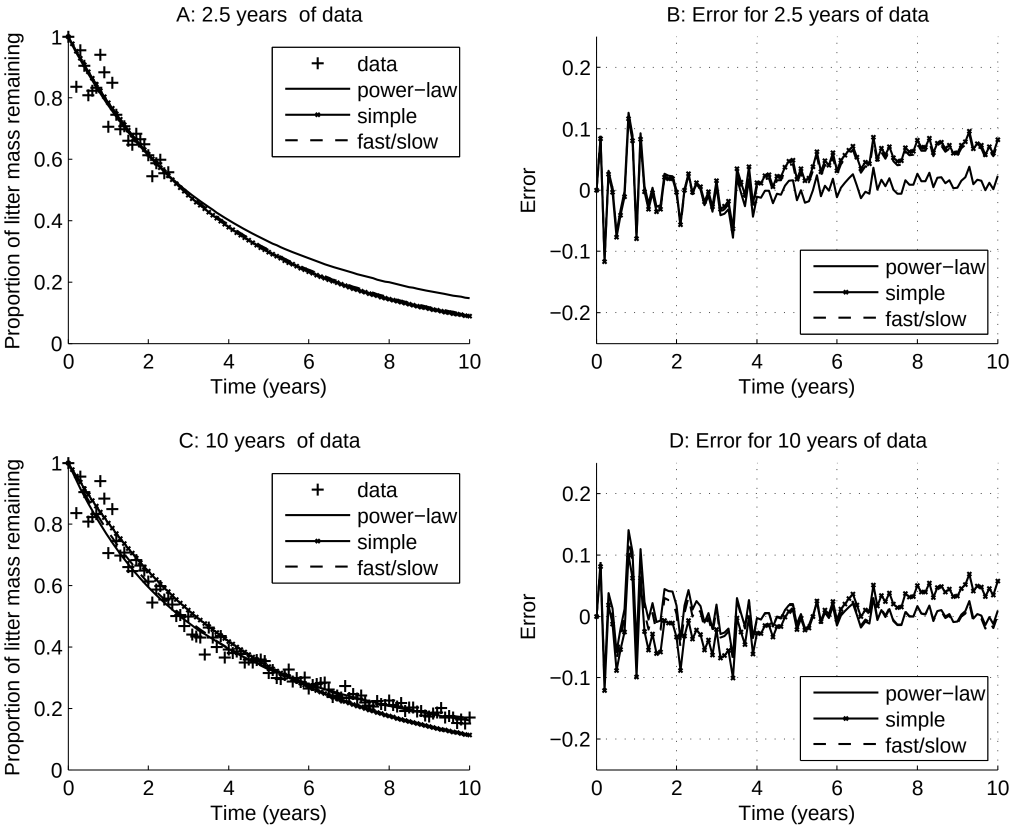

Figure 3: Data, estimated model, and error trajectories for the power-law, two-compartment fast/slow, and simple exponential decay models for 2.5 and 10 years of data with 5% variability.

as can be seen in Table 5. The extremely wide confidence interval for k_{fast} and the 2.5 year data set clearly indicate an ill-conditioned parameter covariance matrix, a consequence of over-fitting a model given the amount of data. In addition, for the 5.0 year data set, the confidence intervals for F_{slow}, k_{slow}, and k_{fast} all contain zero, possibly indicating a slightly ill-conditioned parameter covariance matrix. In fact, k_{slow} is equal to zero to at least four decimal places (actually seven), producing an essentially constant term for the slow decay compartment, a result that is not biologically plausible given that a two compartment model was assumed. The 7.5 year data set also produced a confidence interval for k_{slow} that contained zero, and this interval should have been long enough and contained enough data points to successfully estimate three independent parameters for identifiable compartments in the two-compartment double exponential decay fast/slow decomposition model. The estimated parameters for the 10 year data set were comparable in magnitude to those from the no added variability scenario, but the values were different. The trajectory for the two-compartment double exponential decay fast/slow model with added variability and the 2.5 year data set in Figure 3 was more similar to the simple exponential decay trajectory than the power-law trajectory, and again continued to diverge from the power-law model, approaching zero much more rapidly. The trajectory for the 10.0 year data set was in very good agreement with the power-law trajectory, tracking through the center of the data.

The addition of variability also affected the estimation of the power-law model parameters for the different time intervals: the exact parameter values $\alpha_f = 0.3660$ and $g_f = 2.3365$ were not recovered, but they were not expected to be recovered. The estimated parameter values were, however, consistent as more data were included, and converged to fixed values that produced a trajectory that was close to the actual trajectory. Further, all four of the estimated values of $\hat{\alpha}_f$ were contained in the 95% confidence intervals for each time interval, with decreasing confidence interval widths as more data were added indicating convergence to a stable value. The estimated values of $\hat{g}_f$ behaved similarly, except for the estimated parameter value for the 2.5 year data set which was outside the confidence intervals for the three longer time intervals. The trajectory for the 2.5 year data set fits the data well, but does not fall off as rapidly as the other two models, overestimating the value at 10.0 years by approximately 0.05, while the other models underestimate the final value by approximately 0.15. The power-law trajectory for the 10.0 year data set was in very good agreement with the data, tracking through the center of the data over the entire time interval.

Results of the IAE comparisions for the two assessment scenarios and the three decomposition models are presented in Table 6. The poor performance of the simple exponential decay decomposition model is readily apparent from the computed IAE values, none of which are near zero. The IAE results for the power-law decomposition model and the no variabilty scenario clearly indicate that the true trajectory was recoverd almost exactly, with all four time intervals having IAE values that were zero to more than four decimal places (actually 12). For the added variability scenario, the power-law model begins with an IAE value of 0.2097 for the 2.5 year data set, 0.2271 for the 5.0 year data set, and decreasing values of 0.2002 and 0.1991 for the remaing two time intervals. Comparing the power-law model estimated using the added variability to the true model produced IAE values of 0.11.27, 0.1002, 0.0288, and 0.0150, respectively, for the 2.5 to 10.0 year data sets, giving decreasing IAE values for each addition of data. The slight increase in the IAE value for the 5.0 year data set was most likely a result of several relatively large deviations that occur between years 3 and 4, visible in the 10.0 year trajectory, since similar behavior was not observed when the estimated trajectory for the added variablility scenario was compared with the true trajectory.

The IAE results for the two-compartment double exponential decay fast/slow decomposition model and the no variability scenario indicate that the model fit steadily improves as more data are added, beginning with an IAE value of 0.1206 and decreasing to 0.0314, 0.0111, and 0.0072 for the three longer time intervals, respectively. For the added variability scenario, two-compartment double exponential decay fast/slow model performed 2 times worse for the 2.5 year data set, when compared to the added variability alone for the entire 10 year trajectory, having an IAE value of 0.4208 *vs.* an IAE value of 0.2005. The remaining three time intervals were more reasonable, with IAE values of 0.2075, 0.1987, and 0.1918, indicating slightly worse agreement than the power-law model for the 5.0 year data set and comparable results for the two remaining time intervals. Comparing the two-compartment double exponential decay fast/slow model estimated using added variability to the true model produced IAE values of 0.3508, 0.1092, 0.0718,

Table 6: Estimated IAE values for four time intervals, computed using ten year projections, for the assessment scenarios. The IAE value for the 10.0 year data set with added variability and the true trajectory was 0.2005.

		Time interval			
Model	Scenario	2.5	5.0	7.5	10.0
Power-law	No variability	0.0000	0.0001	0.0000	0.0000
	Added variability	0.2097	0.2271	0.2002	0.1991
	Variability *vs.* true	0.1127	0.1002	0.0288	0.0150
Fast/slow	No variability	0.1206	0.0314	0.0111	0.0072
	Added variability	0.4208	0.2075	0.1987	0.1918
	Variability *vs.* true	0.3508	0.1092	0.0718	0.0628
Simple	No variability	0.5539	0.4006	0.3379	0.3217
	Added variability	0.4556	0.4347	0.3705	0.3529
	Variability *vs.* true	0.3891	0.3733	0.3305	0.3216

and 0.0628, respectively, for the 2.5 to 10.0 year data sets, giving decreasing IAE values for each addition of data. Again, we see a large IAE value for the 2.5 year data set, with more reasonable values for subsequent time intervals, but the IAE values for the two-compartment double exponential decay fast/slow model and the 5.0, 7.5, and 10.0 year data sets had IAE values that were 1.5 to 2 times greater than those for the power-law model when making the same comparisons.

The IAE values for the no variability, added variability, and added variability *vs.* true trajectory provide an indication of how well, or poorly, the estimated model trajectories for each data set agree with the 10 year trajectories that were used to estimate the model parameters or the true trajectory, as well as providing an indication of the potential for the model to perform well for extrapolation outside the range of the available data when combined with a visual assessment. The simple exponential decay model did not fit the data well for any of the time intervals, nor did it perform well when extrapolating beyond the 2.5, 5.0, and 7.5 year data set boundaries. The two-compartment double exponential decay fast/slow model produced relatively large IAE values of 0.1206, 0.4208, and 0.3508 for the no added variability, added variability, and added variability *vs.* true trajectory comparisons indicating that the good initial agreement with the 2.5 year data set did not extend to extrapolation for the remainder of the 10 year trajectory. Agreement for this model did improve as additional data were included, but it still performed worse relative to the power-law model in the added variability comparisons. The power-law model recovered the exact trajectory for the no variability scenario, produced a trajectory from the 2.5 year data set that was in good agreement with the 10 year projection with an IAE value that was on the order of the added variability, 0.2097 *vs.* 0.2005, over the entire 10 year projection interval, indicating the ability to consistently extrapolate beyond the available data. The agreement with the 10 year trajectory also improved as more data were added for each of the remaining time intervals. The power-law model also performed well when the trajectories produced by the model estimated from the added variability scenario were compared with the true trajectory.

The behavior of the two-compartment double exponential decay fast/slow decomposition model that we have just described strongly suggests that the improved fit obtained for this model, relative to the simple exponential decay model, is a regression effect due to over fitting, rather than an indication of a second decomposition compartment for uniform needle litter. We have also demonstrated that the two-compartment fast/slow decomposition model can provide a good fit to the data, but that variability can cause biologically implausible models as identified by the estimated parameter values. We have also shown that the two-compartment fast/slow model does not extrapolate well when used with data collected over short time periods relative to the rate of decomposition. Harmon et al. (1990) did not report confidence intervals for their two-compartment double exponential decay fast/slow model parameter estimates, nor did they examine the behavior of the model when extrapolating beyond the range of the data, so it is not clear whether they encountered the types of numerical issues described here. Given the overall fidelity of the new power-law model to the observed data, its consistency in the parameter estimation process with the addition of new data, its stability when extrapolating beyond the available data, and its smaller number of necessary assumptions, the power-law model should be preferred to the two-compartment double exponential decay fast/slow model, by Occam's Razor, for representing the decomposition process for a uniform material.

4.2. Other data

The power law model, the two-compartment double exponential decay fast/slow model, and the simple exponential decay model the were applied to a variety of other needle/leaf and twig decomposition data sets from Harmon (2013). Results from that analysis were comparable to those obtained here for the Pacific silver-fir data.

5. Conclusions

A new power-law model for the litter mass remaining after decomposition over time has been proposed. The new power-law model outperforms the commonly used simple exponential decay decomposition model, providing a high degree of fidelity to observed data, and it also performs very well when compared with a two compartment double exponential decay fast/slow decomposition model (Bunnell and Tait, 1974). The power-law model introduces no new assumptions, unlike the two-compartment double exponential decay fast/slow model which assumed a second identifiable compartment, and it changes only the mathematical form of the relationship between the current rate of decomposition and the current litter mass, making it a straightforward generalization of the simple exponential decay model. The new power-law model enables the use of a complete time series of data for model identification and parameter estimation, overcoming the primary limitations of the simple exponential decay model, and the power-law model also provides parameter estimates that appear to converge to fixed values with the addition of new data over time, as well as providing consistent extrapolation outside the range of available data.

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